

A Hybrid Model of Neutrino Masses and Oscillations: Bulk Neutrinos in the Split-Fermion Scenario

Keith R. Dienes^{1*}, Sabine Hossenfelder^{1,2†}

¹*Department of Physics, University of Arizona, Tucson, AZ 85721 USA*

²*Department of Physics, University of California, Santa Barbara, CA 93106 USA*

(Dated: July 10, 2006)

Higher-dimensional models of neutrino physics with one or more right-handed neutrinos in the bulk have attracted considerable attention in recent years. However, a critical issue for such models is to find a way of introducing the required flavor dependence needed for generating neutrino oscillations. In this paper, we point out that a natural “minimal” framework that accomplishes this can be constructed by combining the bulk-neutrino hypothesis for right-handed neutrinos with the split-fermion scenario for left-handed neutrinos. This combination leads to a unique flavor signature for neutrino phenomenology which easily incorporates large flavor mixing angles. This hybrid scenario also has a number of additional important features. For example, one previous difficulty of the split-fermion scenario applied to neutrinos has been that the mass matrix is exponentially sensitive to neutrino displacements within the brane. However, in our hybrid scenario, the interactions between the brane and bulk naturally convert this dependence from exponential to linear. Another important feature is that our hybrid scenario provides its own natural regulator for Kaluza-Klein sums. Thus, in our scenario, all Kaluza-Klein summations are manifestly finite, even in cases with multiple extra dimensions. But most importantly, our mechanism completely decouples the effective neutrino flavor mixing angles from the sizes of the overlaps between the neutrino wavefunctions within the brane. Thus, we are able to obtain large neutrino mixing angles even when these neutrinos have significant spatial separations and their overlaps vanish.

I. INTRODUCTION AND OVERVIEW

Many physicists consider neutrino oscillations to be the first clear signature for physics beyond the Standard Model. With the most recent data from SNO and Super-Kamiokande, there remains virtually no doubt about the existence of neutrino flavor oscillations. Indeed, the parameter space for neutrino mass differences and mixing angles has already become significantly constrained.

There remains, however, the paramount issue concerning how these experimentally determined parameters can be accommodated within a theoretically motivated model. While there have been numerous avenues that have been explored within recent years, models with large extra dimensions have received considerable attention ever since it was realized [1, 2, 3, 4, 5] that extra spacetime dimensions have the potential to provide new, intrinsically geometric perspectives on the hierarchies between various energy scales present within and beyond the Standard Model (SM). These include the Planck scale [1, 5], the GUT scale [2], and the string scale [2, 3], as well as a possible scale for SUSY-breaking [4]. Neutrino phenomenology has also been investigated within this higher-dimensional context, with the fundamental idea [6, 7] being that since the right-handed neutrino is a Standard-Model singlet, it need not necessarily be bound to the brane to which the other Standard-Model particles are restricted. The right-handed neutrino can therefore

propagate into the bulk of extra spacetime dimensions and accrue an infinite tower of Kaluza-Klein (KK) excitations, all of which will then take part in neutrino oscillations [7]. This idea, along with numerous variations, has spawned a relatively large literature investigating various aspects of higher-dimensional neutrino phenomenology (see, *e.g.*, Refs. [6, 7, 8, 9]).

The challenge for these bulk-neutrino models, however, is to accommodate a suitable flavor structure which can trigger the observed neutrino oscillations. To date, two different directions have been explored in the literature.

One direction involves generalizing the original idea of Refs. [6, 7] by introducing a separate right-handed bulk neutrino for each flavor of neutrino on the brane. This process thereby extends flavor into the bulk. However, in such cases, the brane/bulk couplings become arbitrary 3×3 mixing matrices whose parameters are in principle undetermined. Moreover, the three bulk neutrinos can in principle correspond to different extra spacetime dimensions with different radii. Thus, one obtains a scenario with many undetermined parameters governing neutrino masses and mixing angles. Indeed, to a large extent, the role of the extra dimensions in such scenarios is reduced to providing a set of Kaluza-Klein states which function (from a four-dimensional perspective) as additional sterile neutrinos. In contrast, flavor neutrino oscillations continue to be triggered through mixing angles which are ultimately introduced by hand, just as in four dimensions. This therefore fails to yield any further insight regarding the structure or origins of the neutrino mixings.

A second, more “minimal” idea first advanced in Ref. [9] is to introduce only one bulk neutrino, and to pro-

*E-mail address: dienes@physics.arizona.edu

†E-mail address: sabine@physics.ucsb.edu

vide this single bulk neutrino with flavor-universal couplings to all three brane neutrinos. Thus, in so doing, one is essentially considering flavor to be a feature internal to the Standard Model, a feature which is restricted to the brane and which therefore does not extend into the bulk. It is even possible to take the brane theory to be flavor-diagonal, thereby avoiding mixing angles completely; indeed, in the model of Ref. [9], the only flavor-sensitive feature that distinguishes these brane neutrinos is their differing bare Majorana masses m_i . Nevertheless, one finds [9] that sizable neutrino flavor oscillations between the different neutrinos on the brane arise as a result of their indirect mixings with the Kaluza-Klein modes of the bulk right-handed neutrino. Thus, in such a model, the flavor oscillations are entirely “bulk-mediated” and no mixing angles are needed at all: it is the presence of the higher-dimensional bulk which is completely responsible for inducing the flavor oscillations on the brane.

Although it is possible to consider cases in which the couplings between the brane neutrinos and the bulk neutrino are large, it is convenient (and perhaps also phenomenologically necessary) to consider a so-called “perturbative” limit in which these couplings are relatively small. In such cases, it is possible to integrate out the effects of the Kaluza-Klein bulk neutrinos in order to obtain an effective mixing matrix involving only the light modes in the theory. Typically, these light modes consist of the three brane neutrinos as well as the zero mode(s) of the bulk neutrino(s). Because this effective mixing matrix completely encapsulates the resulting flavor-dependence of the model, it inevitably lies at the center of any comparison between a given theoretical neutrino model and experimental data. Indeed, experimentalists will eventually provide a unique numerical mixing matrix which incorporates the entire observed phenomenology of neutrino oscillations. It will then be the job of the theorist to solve the “inverse” problem of deducing the set of viable underlying neutrino models which can lead to this matrix.

The non-minimal models with many bulk neutrinos contain many mixing angles and can therefore give rise to relatively diverse effective mixing matrices. However, one difficulty with the minimal model of Ref. [9] is that it leads to a rather rigid form for this effective mixing matrix. Since the minimal model involves only one bulk neutrino, it yields an effective 4×4 mixing matrix which, as we shall see, has a texture of the form

$$\mathcal{M} = \begin{pmatrix} m_1 + X & X & X & m \\ X & m_2 + X & X & m \\ X & X & m_3 + X & m \\ m & m & m & 0 \end{pmatrix} \quad (1)$$

where the first three rows/columns correspond to the brane neutrinos and the final row/column corresponds to the bulk zero mode. Here X and m are parameters associated with the bulk of the higher-dimensional theory and its coupling to the brane, while m_i are the bare Majorana masses of the neutrinos on the brane. While we see

that the bulk physics (through the term X) is responsible for yielding a non-diagonal 3×3 brane mixing submatrix, this matrix is relatively rigid, with all off-diagonal entries forced to be exactly equal. Indeed, a mixing texture of this form may have difficulty accommodating observed neutrino oscillations of the correct sizes and magnitudes.

It is the purpose of this paper to propose an alternative “minimal” model which leads to a richer effective mixing matrix without introducing additional parameters or sacrificing any of the minimality of the single bulk-neutrino scheme. Rather than incorporate a flavor structure through differing bare Majorana masses m_i on the brane as in Ref. [9], we shall instead incorporate a non-trivial flavor structure within the context of the so-called “split-fermion” scenario [10]. As is well known, the split-fermion scenario represents an intrinsically higher-dimensional method of explaining flavor hierarchies for the charged matter content of the Standard Model. However, as we shall discuss below, this method faces certain unique difficulties when attempting to address the flavor structure of the neutrino sector. Thus, by combining the bulk-neutrino scenario with the split-fermion scenario, we are able to obtain a “hybrid” minimal model which has both strong theoretical motivations as well as rich prospects for neutrino phenomenology.

II. SPLIT FERMIONS

In the split-fermion scenario [10], the fermions of the Standard Model are located within a “fat” brane but centered around different positions within the brane. Their spatial extent within the fat brane is modeled by a Gaussian distribution with a typical width σ , where σ is approximately one order of magnitude smaller than the width of the brane. The gauge fields and the Higgs are assumed to be equally distributed over the width of the brane. To obtain the effective four-dimensional couplings between different particles, the extra dimensions have to be integrated out. Since the overlap of the Gaussian wavefunctions of the particles can be very small, the resulting effective couplings can be extremely suppressed. Indeed, it is easy to obtain a suppression by thirty orders of magnitude by delocalizing particles at locations which are separated by about ten times their Gaussian widths.

This setup yields a solution to several problems. It can be used to suppress the proton decay, to suppress flavor-changing operators, or to explain the observed mass hierarchy in the lepton sector. Many possible configurations of particles inside the brane are possible; some of these are discussed in Ref. [11, 12, 13].

Relative shifts between left-handed Standard-Model doublets and right-handed singlets have been successfully used to address the mass hierarchy of quarks and charged leptons. But for the same reason the split-fermion scenario is easy to implement for charged fermions, it leads to difficulties when one tries to incorporate neutrinos [14]. Experiments suggest with an increasing degree of cer-

tainty that neutrino mixing is maximal in the case of $\nu_\tau \leftrightarrow \nu_\mu$ and almost maximal in the case of $\nu_e \leftrightarrow \nu_\tau$. The common textures of the zeroth-order neutrino mass matrix that are compatible with such almost-bimaximal mixing can be classified by the hierarchy type into different categories — normal, inverted, or degenerate. However, all of these situations require several entries to be of the same order of magnitude.

Unfortunately, in the split-fermion scenario, the coupling between any two particles is extremely sensitive to their relative distance within the brane. The observed neutrino mixing can therefore be achieved only by carefully choosing the central locations of the neutrinos inside the brane. In addition, suitable positions for the charged leptons must be found in order to avoid excessive flavor-changing processes. Together, these requirements put severe constraints on the allowed fermion locations [14]. While all experimental constraints can ultimately be accommodated, the resulting particle map is extremely sensitive to small perturbations. Thus, even though these models reduce the amount of fine-tuning for the Yukawa couplings, a considerable degree of fine-tuning continues to be necessary.

There are, of course, various options for ameliorating this situation. For example, introducing Majorana neutrinos rather than Dirac neutrinos has been proposed [15] as a method of achieving reasonable neutrino masses within the split-fermion scenario. The neutrino sector of the split-fermion scenario has also recently been examined [16] within the context of the Randall-Sundrum model [5].

In this paper, we will consider a different approach towards the neutrino sector by asking what consequences arise if the two scenarios — split fermions and bulk neutrinos — are examined in a joined framework. In this way, we shall be combining the strengths of each individual scenario: the split-fermion mechanism will trigger the flavor structure of the neutrino sector, while the bulk sector will ameliorate the fine-tuning issues and naturally lead to large mixing angles. Indeed, it was already noted in Ref. [7] that the combination of having brane neutrinos at different locations in the extra dimension would trigger a higher-dimensional seesaw mechanism, although this idea was not pursued and no consistent framework was provided. In this paper, we shall see that split fermions provide a natural context for this phenomenon, leading to flavor-dependent mixing properties which can be used to distinguish between the brane neutrinos.

There are also other benefits to combining these scenarios. For example, as we have mentioned, one difficulty of the split-fermion scenario applied to neutrinos has been that the mass matrix is exponentially sensitive to neutrino displacement within the brane. However, in our hybrid scenario, we shall see that the interactions between the brane and bulk naturally convert this dependence from exponential to linear. Thus, the previous exponential sensitivity is entirely eliminated. Another important feature is that our hybrid scenario provides its

own natural regulator for Kaluza-Klein sums. Thus, in our scenario, all such Kaluza-Klein summations are manifestly finite, even in cases with multiple extra dimensions. As we shall see, this arises because the heavy Kaluza-Klein modes with wavelengths that are small compared with the widths of the brane Gaussians will average out when folded with the Gaussian distribution. This thereby eliminates the necessity of introducing a cutoff by hand.

But most importantly, we shall find that our mechanism completely decouples the effective neutrino flavor mixing angles from the magnitudes of the neutrino overlaps within the brane. Thus, we are able to obtain large neutrino mixing angles between neutrinos, even when these neutrinos have rather large spatial separations within the brane.

III. THE HYBRID FRAMEWORK

We now describe the framework for our hybrid model. Clearly, parts of this model will be similar to the model considered in Ref. [9]. For concreteness, we will work in this section with one bulk neutrino and one compactified extra dimension of radius R , although the model can be trivially extended to more bulk fields and to more additional dimensions.

The primary features of this model can be summarized as follows. We imagine a single extra dimension of radius R , as well as a brane of width $R' \ll R$.

- On the brane, we introduce n_f left-handed neutrinos ν_α ; these are our flavor eigenstates. We assume that these left-handed neutrinos have no corresponding Majorana masses and are restricted to lie within the brane.
- In the bulk, we introduce a single Dirac fermion Ψ which does not carry any flavor indices and is therefore completely flavor-neutral.
- For the sake of minimality, we assume that the left-handed neutrinos do not mix or couple to each other directly. Thus, all mixing angles on the brane are set to zero.
- We introduce a single flavor-blind brane/bulk coupling M_* between each of the brane neutrinos ν_α and the bulk neutrino Ψ .
- Finally, each of the active brane neutrinos ν_α is assumed to be centered around a different transverse location within the brane. It is this feature, and only this feature, which introduces a flavor-dependence into our model.

We shall take our spacetime coordinates as $x^A \equiv (x^\mu, y)$ where x^μ are the four uncompactified coordinates on the brane and y is the coordinate of the fifth dimension perpendicular to the brane. Given the above assumptions, the action for our model then takes the form

$\mathcal{S} = \mathcal{S}_\nu + \mathcal{S}_b + \mathcal{S}_c + \text{h.c.}$ where \mathcal{S}_ν describes the physics on the brane, \mathcal{S}_b describes the physics in the bulk, and \mathcal{S}_c describes the brane/bulk couplings. These individual contributions take the form

$$\begin{aligned}\mathcal{S}_\nu &= \int d^4x dy \sum_{\alpha=1}^{n_f} \nu_\alpha^\dagger(x, y) i\bar{\sigma}^\mu \partial_\mu \nu_\alpha(x, y) \\ \mathcal{S}_b &= \int d^4x dy \bar{\Psi}(x, y) i\Gamma^A \partial_A \Psi(x, y) \\ \mathcal{S}_c &= \int d^4x dy \sum_{\alpha=1}^{n_f} M_* \nu_\alpha^\dagger(x, y) [\psi_+(x, y) + \bar{\psi}_-(x, y)]\end{aligned}\quad (2)$$

where Γ represent the five-dimensional Dirac matrices. Here the index $\alpha = 1, \dots, n_f$ runs over all n_f flavors on the brane.

In the above, we have chosen to work in the Weyl basis in which the Dirac fermion Ψ can be decomposed into two two-component spinors: $\Psi = (\psi_+, \bar{\psi}_-)^T$. The bar on $\bar{\psi}_-$ indicates that $\bar{\psi}_-$ transforms as a different Lorentz representation than ψ_+ . We refer to ψ_+ and $\bar{\psi}_-$ as left-handed and right-handed components, respectively. We have also written $\sigma^\mu = (1, -\sigma^i)$, $\bar{\sigma}^\mu = (1, +\sigma^i)$ where σ^i are the Pauli matrices.

Given the action in Eq. (2), the next step is to compactify the fifth dimension in order to obtain an effective four-dimensional theory. To do this, we shall make the following assumptions. First, we shall identify $y \approx y + 2\pi R$ and impose the orbifold relations $\psi_+(-y) = \psi_+(y)$ and $\bar{\psi}_-(-y) = -\bar{\psi}_-(y)$, where y is the coordinate of the fifth dimension. Thus, ψ_+ and $\bar{\psi}_-$ have the Kaluza-Klein mode expansions

$$\begin{aligned}\psi_+(x, y) &= \frac{1}{\sqrt{\pi R}} \psi_+^{(0)}(x) \\ &\quad + \frac{1}{\sqrt{2\pi R}} \sum_{n>0} \psi_+^{(n)}(x) \cos\left(\frac{ny}{R}\right) \\ \bar{\psi}_-(x, y) &= \frac{1}{\sqrt{2\pi R}} \sum_{n>0} \bar{\psi}_-^{(n)}(x) \sin\left(\frac{ny}{R}\right).\end{aligned}\quad (3)$$

Note that as a result of our orbifold boundary conditions, $\bar{\psi}_-$ does not have a zero mode. We shall also take our brane neutrinos to have wavefunctions of the form

$$\nu_\alpha(x, y) = G(y - y_\alpha, \sigma) \nu_\alpha(x) \quad (4)$$

where $\nu_\alpha(x)$ denotes the usual four-dimensional part of the fermion wavefunction on our brane and where $G(y, \sigma)$ denotes a normalized Gaussian wavefunction centered around location $y = 0$ with width σ :

$$G(y, \sigma) = \frac{1}{\sqrt{\sigma}} \exp\left(-\frac{\pi y^2}{2\sigma^2}\right). \quad (5)$$

Note that for simplicity, we are assuming that each brane neutrino has the same width $\sigma \ll R$.

Inserting the wavefunctions in Eqs. (3) and (4) into the action (2) and integrating over the fifth dimension, we obtain the effective four-dimensional actions:

$$\begin{aligned}\mathcal{S}_\nu &= \int d^4x \sum_{\alpha=1}^{n_f} \nu_\alpha^\dagger i\bar{\sigma}^\mu \partial_\mu \nu_\alpha \\ \mathcal{S}_b &= \int d^4x \left\{ \bar{\psi}_-^{(0)} i\sigma^\mu \partial_\mu \bar{\psi}_-^{(0)} \right. \\ &\quad + \sum_{n>0} \left[\psi_+^{\dagger(n)} i\bar{\sigma}^\mu \partial_\mu \psi_+^{(n)} + \bar{\psi}_-^{\dagger(n)} i\sigma^\mu \partial_\mu \bar{\psi}_-^{(n)} \right] \\ &\quad \left. + \sum_{n>0} \left(\frac{n}{R}\right) \left[\psi_+^{\dagger(n)} \bar{\psi}_-^{(n)} + \bar{\psi}_-^{\dagger(n)} \psi_+^{(n)} \right] \right\} \\ \mathcal{S}_c &= \int d^4x \sum_{\alpha=1}^{n_f} \nu_\alpha^\dagger \left\{ m \bar{\psi}_-^{(0)} \right. \\ &\quad \left. + \sum_{n>0} \left(m_{n,+}^\alpha \psi_+^{(n)} + m_{n,-}^\alpha \bar{\psi}_-^{(n)} \right) \right\}.\end{aligned}\quad (6)$$

Note that the fields are now functions of x^μ only, and m and $m_{n,\pm}^\alpha$ are volume-suppressed n -dependent brane/bulk couplings resulting from the rescaling of the individual $\psi_+^{(n)}$, $\bar{\psi}_-^{(n)}$ Kaluza-Klein modes:

$$\begin{aligned}m &\equiv M_* \sqrt{\frac{\sigma}{2\pi R}} \\ m_{n,+}^\alpha &\equiv \sqrt{2}m \exp\left(-\frac{n^2 \sigma^2}{2\pi R^2}\right) \cos\left(\frac{n}{R}y_\alpha\right) \\ m_{n,-}^\alpha &\equiv \sqrt{2}m \exp\left(-\frac{n^2 \sigma^2}{2\pi R^2}\right) \sin\left(\frac{n}{R}y_\alpha\right).\end{aligned}\quad (7)$$

As we see, the brane/bulk coupling is suppressed by a volume factor $\sqrt{\sigma/R}$. Moreover, for non-shifted brane neutrinos with $y_\alpha = 0$, we see that the couplings $m_{n,-}^\alpha$ actually vanish. Thus, it is only the possible displacement of the antisymmetric fermion modes $\bar{\psi}_-^{(n)}$ away from $y_\alpha = 0$ which permits them to couple to the bulk zero mode.

Given the Lagrangian in Eq. (6), we immediately see that the SM flavor-eigenstate neutrinos ν_α on the brane will mix with the entire tower of Kaluza-Klein states of the higher-dimensional Ψ field, even though they do not mix directly with each other. Defining

$$\mathcal{N}^T \equiv (\nu_\alpha, \psi_+^{(0)}, \psi_+^{(1)}, \bar{\psi}_-^{(1)}, \psi_+^{(2)}, \bar{\psi}_-^{(2)}, \dots), \quad (8)$$

we see that the mass term of the Lagrangian (6) takes the form $\mathcal{N}^T \mathcal{M} \mathcal{N}$ where the mass matrix \mathcal{M} is given by

$$\mathcal{M} = \begin{pmatrix} 0 & m & \dots & m_{n,+}^\alpha & m_{n,-}^\alpha & \dots \\ m^T & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ m_{n,+}^{\alpha T} & 0 & \dots & 0 & n/R & \dots \\ m_{n,-}^{\alpha T} & 0 & \dots & n/R & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (9)$$

Of course, in this notation, the first row/column represents n_f independent rows/columns, one for each flavor neutrino on the brane. The fact that this mass matrix is non-diagonal implies that each of the n_f brane neutrinos will undergo oscillations with the bulk KK modes. Through these mixings with the bulk KK neutrinos, the brane neutrinos will thereby undergo effective flavor oscillations with each other.

The next step is to determine the eigenvalues and eigenvectors of the mass matrix (9). We can begin this task as follows. Defining the eigenvector $v = (v^\alpha, v^{(0)}, v_+^{(1)}, v_-^{(1)}, \dots, v_+^{(n)}, v_-^{(n)}, \dots)$, we see that the eigenvalue equation $\mathcal{M}v = \lambda v$ yields the simultaneous equations:

$$\begin{aligned} mv^{(0)} + \sum_{n>0} \left(m_{n,+}^\alpha v_+^{(n)} + m_{n,-}^\alpha v_-^{(n)} \right) &= \lambda v^\alpha \\ m \sum_\alpha v^\alpha &= \lambda v^{(0)} \\ \sum_\alpha m_{n,+}^\alpha v^\alpha + \frac{n}{R} v_-^{(n)} &= \lambda v_+^{(n)} \\ \sum_\alpha m_{n,-}^\alpha v^\alpha + \frac{n}{R} v_+^{(n)} &= \lambda v_-^{(n)}. \end{aligned} \quad (10)$$

Combining the last two of these equations yields a partial solution for $v_\pm^{(n)}$,

$$v_\pm^{(n)} = \frac{1}{\lambda^2 - (n/R)^2} \sum_\alpha v^\alpha \left(m_{n,\pm}^\alpha \lambda + m_{n,\mp}^\alpha \frac{n}{R} \right), \quad (11)$$

and inserting this into the top line of Eq. (10) along with the definitions in Eq. (7) yields the relation

$$\begin{aligned} \lambda v^\alpha &= mv^{(0)} + 2m^2 \sum_{n>0} \frac{\exp\left(-\frac{n^2 \sigma^2}{2\pi R^2}\right)}{\lambda^2 - (n/R)^2} \times \\ &\sum_\beta v^\beta \left(\lambda \cos\left[\frac{n}{R}(y_\alpha - y_\beta)\right] + \frac{n}{R} \sin\left[\frac{n}{R}(y_\alpha + y_\beta)\right] \right) \end{aligned} \quad (12)$$

Thus, we see that the task of solving the remaining equations has now been reformulated as an equivalent $(n_f + 1) \times (n_f + 1)$ -dimensional eigenvalue problem of the form

$$\widetilde{\mathcal{M}}(\lambda) \tilde{v} = \lambda \tilde{v} \quad (13)$$

where $\tilde{v} = (v^\alpha, v^{(0)})$ and where $\widetilde{\mathcal{M}}(\lambda)$ is a λ -dependent “reduced” mass matrix of the form

$$\widetilde{\mathcal{M}}(\lambda) = \begin{pmatrix} m_{\alpha\beta}(\lambda) & m \\ m^T & 0 \end{pmatrix}. \quad (14)$$

Here $m_{\alpha\beta} \equiv (M_*^2 \sigma / \pi) f_{\alpha\beta}$, where the dimensionless coefficients $f_{\alpha\beta}(\lambda)$ are given by

$$\begin{aligned} f_{\alpha\beta}(\lambda) &= \sum_{n>0} \frac{\exp\left(-\frac{n^2 \sigma^2}{2\pi R^2}\right)}{\lambda^2 R^2 - n^2} \times \\ &\left(\lambda R \cos\left[\frac{n}{R}(y_\alpha - y_\beta)\right] + n \sin\left[\frac{n}{R}(y_\alpha + y_\beta)\right] \right) \end{aligned} \quad (15)$$

Note that for numerical purposes, Eq. (13) provides a particularly useful simplification of the original infinite-dimensional eigenvalue problem. Solving Eq. (13) for λ and \tilde{v} and inserting these results into Eq. (11) then enables one to obtain the full infinite-dimensional eigenvector v .

Thus far, we have not made any approximations. However, in order to proceed analytically, we shall now take the so-called “perturbative” limit $mR \ll 1$ in which the brane/bulk coupling is small compared with the compactification scale. Introducing this limit is one way of ensuring that the brane sector of our theory (or, as we shall see, a subset of that sector) will experience little or no net loss of probability into the “sterile” tower of bulk excitations. In this limit (see, *e.g.*, Ref. [7]), the eigenvalues of the excited KK neutrino modes will be approximately $\pm n/R$, corresponding to the bulk/bulk sector of the mixing matrix, while the remaining $n_f + 1$ eigenvalues will be much smaller than the masses of the excited modes. Specifically, we see from the form of the reduced mass matrix in Eq. (14) that the eigenvalues of these remaining light modes will scale either as $\lambda \sim M_*^2 \sigma$ or as $\lambda \sim m$. However, the assumption that $mR \ll 1$ implies that $\lambda R \ll 1$ in either case: if $\lambda \sim m$ then clearly $\lambda R \sim mR \ll 1$, while if $\lambda \sim M_*^2 \sigma$ then

$$\lambda R \sim M_*^2 \sigma R \sim (mR)^2 \ll 1. \quad (16)$$

Thus in either case we can approximate

$$f_{\alpha\beta} \approx - \sum_{n>0} \frac{1}{n} \exp\left(-\frac{n^2 \sigma^2}{\pi R^2}\right) \sin\left[\frac{n}{R}(y_\alpha + y_\beta)\right] \quad (17)$$

for these remaining light modes. As we shall demonstrate in Appendix A, this result may also be derived through a direct perturbative diagonalization of our original mass matrix.

Note that the only case in which it is not justified to pass from Eq. (15) to Eq. (17) occurs when y_2 is exactly equal to $-y_1$, for in this case the second term in Eq. (15) vanishes exactly for all n and is thus always smaller than the first. However, since λ is significantly suppressed for the light modes we are considering, we see that the resulting value of $f_{\alpha\beta}$ in this case is significantly smaller than it is for other values of y_1 and y_2 . The approximation in Eq. (17), which sets this value of $f_{\alpha\beta}$ to zero, therefore continues to be roughly accurate even in this case, indicating a strongly suppressed value for $f_{\alpha\beta}$ in the region near $y_2 = -y_1$.

It is relatively straightforward to evaluate the KK sum in Eq. (17), especially since the presence of the Gaussian width for the brane neutrinos serves as a natural regulator for the KK sum which renders it manifestly finite. Introducing the variable $k = (\sigma/R)n$ and noting that $\sigma \ll R$, we see that the sum can be approximated as an integral, yielding

$$f_{\alpha\beta} \approx - \int_0^\infty \frac{dk}{k} \exp(-k^2/\pi) \sin[k(y_\alpha + y_\beta)/\sigma]$$

$$= -\frac{\pi}{2} \text{Erf} \left[\frac{\sqrt{\pi}}{2\sigma} (y_\alpha + y_\beta) \right]. \quad (18)$$

Note that the error function is an odd function, $\text{Erf}(-x) = -\text{Erf}(x)$, with approximate behavior

$$\text{Erf}(x) \approx \begin{cases} x & \text{for } |x| \lesssim 1 \\ \text{sign}(x) & \text{for } |x| \gtrsim 1. \end{cases} \quad (19)$$

Thus, defining the rescaled (dimensionless) displacements $\tilde{y}_\alpha \equiv \sqrt{\pi} y_\alpha / (2\sigma)$, we conclude that for the light modes in our theory, our reduced $(n_f + 1) \times (n_f + 1)$ -dimensional mass matrix in Eq. (14) takes the form

$$\tilde{M} \approx \begin{pmatrix} m_{\alpha\beta} & m \\ m^T & 0 \end{pmatrix} \quad (20)$$

with

$$m_{\alpha\beta} \approx -\frac{M_*^2 \sigma}{2} \times \begin{cases} \tilde{y}_\alpha + \tilde{y}_\beta & \text{for } |\tilde{y}_\alpha + \tilde{y}_\beta| \lesssim 1 \\ \text{sign}(\tilde{y}_\alpha + \tilde{y}_\beta) & \text{for } |\tilde{y}_\alpha + \tilde{y}_\beta| \gtrsim 1. \end{cases} \quad (21)$$

We see from Eq. (21) that when the neutrino displacements are of the same order as their Gaussian widths, the entries of this mass matrix depend *linearly* rather than exponentially on the displacements! This is an important observation, implying that the usual exponential sensitivity to the fermion displacements has been eliminated in our hybrid scenario, with the interaction with the bulk neutrino serving to convert this sensitivity from exponential to linear. Moreover, we also observe the interesting fact that the interactions with the bulk neutrino actually remove *all* sensitivity to the neutrino displacements when these displacements become significantly larger than their Gaussian widths, with mass-matrix values saturating near the universal value $m_{\alpha\beta} \approx -\frac{1}{2} M_*^2 \sigma$ in such cases. Thus, in such cases, all flavor dependence is lost.

We also notice from Eq. (21) that our effective mixings on the brane are not translationally invariant within the brane, since they depend not on the difference $y_\alpha - y_\beta$, but rather on the *sum* $y_\alpha + y_\beta$. Indeed, our off-diagonal mixing terms cancel only for $y_\alpha = -y_\beta$ rather than $y_\alpha = y_\beta$. Moreover, since this dependence is on the sum rather than the difference, the net distance from the origin at $y = 0$ becomes an important quantity, determining when the error function passes from its linear to constant regime. Once again, however, these features represent the effects of the mixings with the bulk neutrino modes, for it is the KK orbifold relations just above Eq. (3) which implicitly select the origin $y = 0$ as a special point, thereby breaking translational invariance within the brane.

These features imply that it is possible to obtain large effective flavor mixing angles for the active neutrinos on the brane *even when the overlaps between these neutrinos vanish*. In other words, large neutrino overlaps are no longer needed for large effective mixing angles.

To illustrate this behavior more concretely, let us consider a simplified two-flavor example in which we have

only one effective mixing angle. Given the result in Eq. (21), we may identify this effective mixing angle θ through the relation

$$\tan 2\theta = \frac{2 \text{Erf}(\tilde{y}_1 + \tilde{y}_2)}{\text{Erf}(2\tilde{y}_2) - \text{Erf}(2\tilde{y}_1)}. \quad (22)$$

For a given fixed value of \tilde{y}_1 (the position of the first neutrino), we can now examine how this effective mixing angle depends on \tilde{y}_2 (the position of the second neutrino).

There are several cases to consider. If $|\tilde{y}_1| \ll 1$ (so that the first neutrino is centered very close to the origin), we find that $\tan 2\theta \rightarrow 2$ as $\tilde{y}_2 \rightarrow \pm\infty$. Thus, we find that $\sin^2 2\theta \rightarrow 4/5$ as $\tilde{y}_2 \rightarrow \pm\infty$, which represents nearly maximal mixing. Note that this result applies even though the second neutrino is being pulled relatively far from the first. Indeed, one can verify the mixing remains nearly maximal in this case *regardless* of the value of \tilde{y}_2 .

By contrast, if $|\tilde{y}_1| \gtrsim 1$, our asymptotic behavior changes. For concreteness, let us take \tilde{y}_1 positive (otherwise we can change all $y \rightarrow -y$ without altering the results). In this case, $\text{Erf}(\tilde{y}_1) \approx 1$ and we see from Eq. (22) that $\tan 2\theta \rightarrow \infty$ as $\tilde{y}_2 \rightarrow \infty$ while $\tan 2\theta \rightarrow 2$ as $\tilde{y}_2 \rightarrow -\infty$. Thus, asymptotically, we have relatively large mixing angles in either case. However, near $\tilde{y}_2 \approx -\tilde{y}_1$, we find that our mixing angle passes through zero and changes sign. Thus, there is a region near $\tilde{y}_2 \approx -\tilde{y}_1$ for which smaller mixing angles can be realized without exponential fine-tuning.

Finally, for $|\tilde{y}_1| \lesssim 1$, the resulting behavior interpolates between the two behaviors described above. Again taking $\tilde{y}_1 > 0$ for simplicity, we find that $\sin^2 2\theta$ asymptotes to values between $1/2$ and $4/5$ as $\tilde{y}_2 \rightarrow -\infty$, and between $4/5$ and 1 as $\tilde{y}_2 \rightarrow +\infty$. However, as in all cases, the effective mixing angle hits zero at $\tilde{y}_2 = -\tilde{y}_1$.

This behavior is shown in Fig. 1 for $\tilde{y}_1 = 0, 1, 2, 3$. In each case, we observe that large mixing angles dominate, arising in most regions of the parameter space. Nevertheless, we see that small and intermediate mixing angles are easy to achieve without significant fine-tuning in the region near $\tilde{y}_2 = -\tilde{y}_1$.

As indicated in the Introduction, the flavor mixing in this model is significantly different from that of an earlier model [9] in which all brane neutrinos are located at $\tilde{y} = 0$ with zero width, distinguished only through their different “bare” Majorana masses m_i on the brane. In such a model, the brane neutrinos couple to only the even bulk modes $\psi_+^{(n)}$, whereupon we see (following the same analysis as performed here) that the effective $(n_f + 1) \times (n_f + 1)$ “reduced” mixing matrix $\tilde{\mathcal{M}}(\lambda)$ takes the form shown in Eq. (1), where

$$X = -\frac{m^2}{\lambda} + \pi m^2 R \cot(\pi R \lambda). \quad (23)$$

The important point here is that X is completely flavor-independent. Thus all of the off-diagonal entries in the effective mass matrix are identical, leading to a relatively rigid flavor structure.

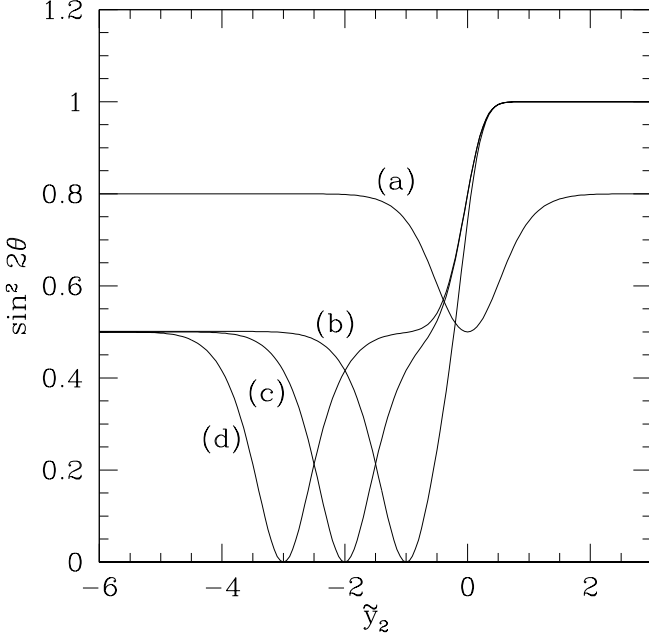


FIG. 1: Effective flavor mixing angle $\sin^2 2\theta$ as a function of neutrino positions $(\tilde{y}_1, \tilde{y}_2)$ in the case of two active flavors on the brane. We plot $\sin^2 2\theta$ as a function of \tilde{y}_2 for $\tilde{y}_1 = 0, 1, 2, 3$ [curves (a) through (d) respectively]. In each case, we see that large mixing angles are fairly common, with small mixing angles emerging only near $\tilde{y}_2 \approx -\tilde{y}_1$.

Thus far, we have focused on the case with only a single extra spacetime dimension. It is therefore worth understanding how the relevant mass scales in the mass matrix (20) are altered if we consider the analogous situation with $\delta > 1$ extra dimensions. With δ extra dimensions, our three-brane has a “width” in δ directions, so that our brane neutrinos are δ -dimensional Gaussians. Tracing our previous steps, this implies that the brane/bulk coupling m defined in Eq. (7) will accrue a further volume factor of $\sqrt{\sigma/R}$ for each additional spacetime dimension, so that we can generally replace

$$m \rightarrow m(\sigma/R)^{(\delta-1)/2} \sim M_*(\sigma/R)^{\delta/2}. \quad (24)$$

On the other hand, it is shown in Appendix B that the entries of $m_{\alpha\beta}$ continue to scale as $M_*^2\sigma$ regardless of the number or radii of extra dimensions. Thus, for general δ , we obtain an effective $(n_f + 1) \times (n_f + 1)$ -dimensional mass matrix whose entries scale as

$$\tilde{M} \sim M_* \begin{pmatrix} M_*\sigma f_{\alpha\beta} & (\sigma/R)^{\delta/2} \\ (\sigma/R)^{\delta/2} & 0 \end{pmatrix} \quad (25)$$

where $f_{\alpha\beta}$ denotes a δ -dependent set of flavor-dependent coefficients of order one. We shall not pursue a detailed calculation of these coefficients for $\delta > 1$, and shall instead rely on our previous results for $\delta = 1$ as a general

illustration of the potential possibilities for the flavor dependence.

These observations are especially relevant for determining the neutrino mass eigenvalues resulting from our mass matrices. Indeed, one of the unique features of these higher-dimensional scenarios involving right-handed neutrinos in the bulk is that it is the *same* mass matrix that determines not only the neutrino flavor mixings but also the neutrino mass eigenvalues. This is different from the traditional four-dimensional framework in which couplings to heavy right-handed neutrinos determine the neutrino masses (through the traditional seesaw mechanism), whereas mixings amongst the light left-handed neutrinos are primarily responsible for determining the effective flavor mixing angles.

It turns out that the eigenvalue phenomenology is very different depending on whether $\delta = 1$ or $\delta > 1$. For $\delta = 1$, we have the mass matrix given in Eq. (20). However, in this case the effective brane/bulk coupling m is necessarily larger than the effective brane/brane components $m_{\alpha\beta}$ by a factor $(mR)^{-1} \gg 1$. Thus, the mixing with the bulk zero mode is dominant and we must actually invert this $(n_f + 1)$ -dimensional mass matrix in order to examine its mass eigenvalues. We find that in general, $n_f - 1$ eigenvalues will be of (the smaller) size $M_*^2\sigma$, while the remaining two eigenvalues will be of (the larger) size m ; likewise, the eigenstates corresponding to $\lambda \sim M_*^2\sigma$ will have an extremely small component involving the right-handed bulk neutrino, while the eigenstates corresponding to $\lambda \sim m$ will have a fairly significant component involving the right-handed bulk neutrino, with probability $\approx \frac{1}{2}$. Thus, the $n_f - 1$ lightest neutrino components are easily interpretable as our light flavor neutrino mass eigenstates — with significant flavor mixings amongst them — provided we choose n_f to *exceed* our eventual desired number of light neutrinos by one.

Given these results, we see that our scenario for $\delta = 1$ actually requires an *additional* left-handed neutrino on the brane which participates in the flavor oscillations but is otherwise exceedingly heavy. In some sense, this neutrino may be regarded as an additional “sterile” neutrino, since its flavor index need not correspond to any of the flavors exhibited by the charged fermions of the Standard Model; moreover, by choosing an appropriate location for this neutrino, its flavor mixings with the other left-handed neutrinos can be minimized and/or eliminated. However, the unique prediction of this model (as opposed to other higher-dimensional models) is that this sterile neutrino is necessarily *left-handed*, living on the brane. Of course, its mass is of the same scale ($\sim m$) as that of the right-handed zero-mode neutrino living in the bulk. Note that the remaining KK bulk excitations have masses scaling as $\sim n/R$, and thus their masses are even heavier by an additional hierarchical factor of $(mR)^{-1} \gg 1$.

For $\delta > 1$, by contrast, the brane/bulk coupling m is suppressed by additional factors of $(\sigma/R) \ll 1$ relative to the brane/brane components $m_{\alpha\beta}$. Thus, assuming that $\sigma/R \ll (mR)^2$, we see that the brane/bulk coupling

will be extremely *small* compared to the brane/brane components, leading to n_f eigenvalues of mass $\sim M_*^2 \sigma$ and a single eigenvalue of mass $\sim \sigma/R^2$. Although this single mass eigenstate will be much lighter than the n_f active eigenstates, it will be overwhelmingly sterile and thus will not interact with the Standard-Model particles.

We see, then, that regardless of the number of extra spacetime dimensions, our active neutrino mass eigenstates will have masses which scale as $\sim M_*^2 \sigma$ in the current setup. There are therefore a number of values for M_* and σ which can yield neutrino masses in a phenomenologically acceptable range. One appealing possibility, for example, is to take M_* (the fundamental mass scale in our theory) of order $M_* \sim \mathcal{O}(\text{TeV})$, while our three-branes have widths of order $\sigma \sim \mathcal{O}(M_{\text{Planck}}^{-1})$. By contrast, the only solution for which M_* and σ^{-1} are of the same order of magnitude requires that we take M_* in the sub-eV range. Likewise, if we demand $\sigma \sim \mathcal{O}(\text{TeV}^{-1})$, then we must take $M_* \sim \mathcal{O}(\text{keV})$ – $\mathcal{O}(\text{MeV})$. However, there are other ways of adjusting the overall mass eigenvalues in this scenario without affecting the flavor mixing angles. For example, if there are two extra dimensions whose coordinates are y and z , we can easily imagine that the Standard-Model brane is located at $z = z_1$ while the right handed neutrino has bulk excitations in the y -direction but is localized around $z = z_2$. If both localizations have common width σ , we can retrace our previous steps to find that this location “mismatch” in the second extra dimension has the effect of inserting an extra suppression factor

$$\exp \left[-\frac{\pi}{2} \frac{(z_1 - z_2)^2}{\sigma^2} \right] \quad (26)$$

for each entry in the resulting effective mass matrix \mathcal{M} . While such an overall rescaling of the mass matrix does not affect the magnitudes of the flavor mixing angles, it directly suppresses the magnitudes of the mass eigenvalues. Thus, in this way, we are free to adjust our mass eigenvalues as needed without disturbing the flavor structure of our model. Other similar scenarios are also possible.

IV. DISCUSSION

In this paper, we have presented a “hybrid” model in which the split-fermion scenario and the bulk-neutrino scenario are joined together. As we have seen, this joining has succeeded in yielding a rich spectrum of flavor mixing angles without significant fine-tuning, thereby ameliorating some of the difficulties inherent in either approach alone.

Needless to say, we have pursued only the most “minimal” approach towards constructing this hybrid model. Further possibilities that we have not exploited include, for example, possible variable widths of the Gaussians [17], the possibility of several right-handed neutrinos, possible orbifold twists [12], or possible additional

flavor-dependent Yukawa couplings of order one. We have also avoided primordial flavor mixings on the brane, and we have taken our brane neutrinos to have vanishing Majorana masses. While all of these assumptions can be relaxed, our main purpose in this paper has been to illustrate the range of theoretical possibilities that emerge solely from the joining of the split-fermion and bulk-neutrino models.

Likewise, we have not performed a detailed fit of the parameters involved, nor have we tried to embed this neutrino sector of the Standard Model within a larger split-fermion framework that also involves the quarks and charged leptons. However, we caution that in order to investigate whether such a hybrid model can accommodate all of the observed neutrino-oscillation properties, it is generally not sufficient merely to focus on reproducing the preferred mass differences and mixing angles often quoted in the neutrino literature. This is because such neutrino mixing parameters are usually quoted within the theoretical context of a simple two- or three-state neutrino mixing. In higher-dimensional neutrino models, by contrast, one usually faces very complex, multi-component neutrino oscillations with a variety of varying oscillation lengths and magnitudes (see, *e.g.*, Ref. [9] for examples of the complexities involved). Indeed, even in the small-coupling limit $mR \ll 1$, one typically has a four-neutrino (or multi-neutrino) oscillation, depending on the number of bulk neutrinos which are present. Thus, in order to perform a proper comparison with experiment, one must go back to the original oscillation data, plotted in terms of appropriate baselines, and perform a direct fit between predicted and observed oscillation patterns (and suitable time-averages thereof) without regard for pre-existing theoretical prejudices. This too represents a direction for further investigation.

There are also several important theoretical issues which we have not examined in this paper. One pressing issue, for example, concerns how such hybrid settings can arise on a fundamental level. Although mechanisms of trapping fermions inside a fat brane have been examined closely, and although various ways to realize a split-fermion scenario seem to be possible, it would be interesting to have a more complete picture in which the required fermion locations can be explained from some deeper principle and/or generated dynamically.

Nevertheless, we have seen in this paper that joining the split-fermion and bulk-neutrino scenarios has produced a rich theoretical model which naturally gives rise to large neutrino flavor mixing angles without significant fine-tuning. Indeed, it is the coupling to the KK modes of the bulk neutrino which eliminates the usual fine-tuning of the split-fermion scenario, and which produces the relatively large mixing angles that we have observed. Thus, within the context of the split-fermion scenario, we have a natural explanation for the somewhat surprising fact that quark mixing angles are generally small while lepton mixing angles are apparently large: while the lepton sector contains a right-handed neutrino which is a gauge

singlet and can therefore propagate in the bulk, all of the fields of the corresponding quark sector carry gauge charges and thus must live within the brane. Thus only the lepton sector is capable of receiving the enhancement to the flavor mixing angles that comes indirectly from the couplings to the KK modes of the right-handed neutrino. We believe that this observation is critical, and perhaps provides one of the strongest motivations for considering such higher-dimensional models of neutrino phenomenology.

Acknowledgments

This work is supported in part by the National Science Foundation under Grant PHY/0301998, by the Department of Energy under Grant DE-FG02-04ER-41298, and by a DFG grant from the German government. We are happy to thank Z. Chacko, H.-S. Goh, and S. Scherer for discussions.

APPENDIX A: PERTURBATIVE DIAGONALIZATION OF THE MASS MATRIX

The neutrino mass matrix \mathcal{M} shows a typical structure which is common for seesaw scenarios. We can therefore analyze some of its features by similar means. Following standard treatments, let us write \mathcal{M} in the block form

$$\mathcal{M} = \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} \quad (27)$$

where the M_L block corresponds to the brane neutrinos as well as the zero mode of the bulk neutrino, where the M_R block corresponds to the excited KK modes of the bulk neutrino, and where M_D corresponds to the couplings between these two groups of states. Assuming that these couplings are small (*i.e.*, assuming $mR \ll 1$, so that we are in the “perturbative” regime), we can then approximately block-diagonalize this matrix by finding a matrix U such that

$$U^T \mathcal{M} U = \begin{pmatrix} \widetilde{M}_L & 0 \\ 0 & \widetilde{M}_R \end{pmatrix}. \quad (28)$$

In general, the solution for U is given by

$$U = \begin{pmatrix} 1 & \kappa \\ -\kappa^T & 1 \end{pmatrix} \quad \text{with } \kappa \approx M_D M_R^{-1}, \quad (29)$$

whereupon we find

$$\widetilde{M}_L \approx M_L - M_D M_R^{-1} M_D^T, \quad \widetilde{M}_R \approx M_R. \quad (30)$$

Since our original mass matrix is real and symmetric, the eigenvalues are necessarily real.

The next-order corrections have the form $M_D M_R^{-1} M_D^T M_D M_R^{-1}$, and an inspection shows that

these are suppressed by a factor of mR relative to those considered above. Likewise, the mixing between the excited KK modes and the brane/zero-mode subsector is suppressed by a factor of $m_{n,\pm}^\alpha R \ll 1$.

Note that in the above, we have chosen to group our blocks in such a way that the zero mode of the bulk neutrino is joined with the brane neutrinos rather than with the bulk-neutrino excited KK states. This is because the KK zero mode is the only mode from the KK tower which is not heavy, and which therefore fails to decouple. By arranging our blocks in this manner, we are therefore “integrating out” only the excited KK modes but retaining the zero mode in our low-energy reduced mass matrix \widetilde{M}_L .

Given the parameters in our specific model, a straightforward calculation gives

$$\kappa = \begin{pmatrix} m_-^{\alpha T} & m_+^{\alpha T} \\ 0 & 0 \end{pmatrix} \frac{R}{n}, \quad (31)$$

whereupon we find

$$\begin{aligned} \kappa M_D^T &= \begin{pmatrix} m_-^{\alpha T} & m_+^{\alpha T} \\ 0 & 0 \end{pmatrix} \frac{R}{n} \begin{pmatrix} m_+^\beta & 0 \\ m_-^\beta & 0 \end{pmatrix} \\ &= \begin{pmatrix} -m_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (32)$$

with

$$m_{\alpha\beta} = - \sum_{n=1}^{\infty} \left(m_{n,+}^{\alpha T} \frac{R}{n} m_{n,-}^\beta + m_{n,-}^{\alpha T} \frac{R}{n} m_{n,+}^\beta \right). \quad (33)$$

We thus find

$$\widetilde{M}_L \approx \begin{pmatrix} m_{\alpha\beta} & m \\ m^T & 0 \end{pmatrix}, \quad (34)$$

whereupon inserting the definitions in Eq. (7) yields the result given in Eq. (17).

APPENDIX B: MASS SCALES FOR $\delta > 1$

We now consider the mass scales involved in our mass matrix \mathcal{M} . We are particularly interested in the mass scales associated with the expression $m_{\alpha\beta}$ in Eq. (33). Thus, there are three components of this expression that we need to generalize to δ dimensions: we need to generalize the couplings $m_{n,\pm}^\alpha$; we need to generalize expressions such as $1/n$; and we need to generalize the KK sum. We shall therefore begin by addressing each of these in turn.

For excitations of the bulk field in more than one extra spacetime dimension, the Kaluza-Klein expansion in Eq. (3) will now involve products of sines and cosines. This in turn means that the couplings $m_{n,\pm}^\alpha$ in Eq. (7) will also be products of sines and cosines whose arguments display the positions y_i of our fields.

Likewise, the form of the higher-dimensional Clifford algebra implies that a Dirac spinor in $4 + \delta$ spacetime dimensions will have $2^{2+\delta/2}$ components, and the inverse of the wave vector n will be a linear combination of its components. However, since our goal is merely to estimate relevant mass scales, we shall ignore these spinorial details and retain a notation whereby we treat quantities such as $1/n$ as a single number.

Finally, we observe that the KK sum in Eq. (33) will now extend over all δ -dimensional vectors $n = (n_1, n_2, \dots, n_\delta)$ with $n_i \in \mathbb{Z}^+$ except the zero vector.

Let us now put these ingredients together. First, we notice that in the higher-dimensional case with $\delta > 1$, the couplings from Eq. (3) will enter into Eq. (15) only for mixtures of even and odd fields. This means that the factors that must be summed as part of the n -summation will consist of at least one sine function and one cosine function. Since the contributions to this sum are naturally damped beyond $n \sim R/\sigma$, and since we assume the locations y_α to be of order σ , we can approximate each occurrence of the sine function by the linear term in its Taylor expansion (and set each cosine factor to 1). Let s denote the number of sine functions in the product, where $1 < s < \delta$, and likewise let $i = 1, \dots, s$ denote those directions for which these sine functions appear. Each of these sine functions will then make a linear contribu-

tion to $m_{\alpha\beta}$ of the form $y_i n/R$, whereupon we find that Eqs. (13) and (15) will generalize to yield an expression of the form

$$m_{\alpha\beta} \sim m^2 R \sum_n \frac{1}{n} \exp\left(-\frac{n^2 \sigma^2}{\pi R^2}\right) \prod_{i=1}^s \left(y_i \frac{n}{R}\right). \quad (35)$$

Note that this n -summation continues to represent a sum over δ -dimensional vectors n . Defining $k \equiv (\sigma/R)n$, we see that this sum can be approximated by an integral

$$m_{\alpha\beta} \sim \frac{m^2 R^\delta}{\sigma^{\delta-1}} \prod_i (y_i/\sigma) \int dk k^{\delta+s-2} \exp(-k^2/\pi^2). \quad (36)$$

However, the integral is now purely dimensionless and $\sim \mathcal{O}(1)$; likewise, for $y_i \sim \sigma$, we see that each of the product factors are also $\sim \mathcal{O}(1)$. Thus the overall mass scale is set by the prefactor in Eq. (36). However, given the result in Eq. (24), we see that we can write this prefactor in the form $M_*^2 \sigma$, obtaining the same overall mass scale as we found for $\delta = 1$. Thus, we conclude that the overall mass scale for $m_{\alpha\beta}$ is independent of both R and δ . Indeed, it is easy to verify that this result remains true even when all of the proper numerical and spinorial factors are included in the analysis.

-
- [1] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998); Phys. Rev. D **59**, 086004 (1999); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998).
 - [2] K. R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. B **436**, 55 (1998); Nucl. Phys. B **537**, 47 (1999); arXiv:hep-ph/9807522.
 - [3] E. Witten, Nucl. Phys. B **471**, 135 (1996); J. D. Lykken, Phys. Rev. D **54**, 3693 (1996); G. Shiu and S. H. H. Tye, Phys. Rev. D **58**, 106007 (1998).
 - [4] I. Antoniadis, Phys. Lett. B **246**, 377 (1990); I. Antoniadis, C. Munoz and M. Quiros, Nucl. Phys. B **397**, 515 (1993); I. Antoniadis and K. Benakli, Phys. Lett. B **326**, 69 (1994).
 - [5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); Phys. Rev. Lett. **83**, 4690 (1999).
 - [6] N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali and J. March-Russell, Phys. Rev. D **65**, 024032 (2002).
 - [7] K. R. Dienes, E. Dudas and T. Gherghetta, Nucl. Phys. B **557**, 25 (1999).
 - [8] See, *e.g.*, G. R. Dvali and A. Y. Smirnov, Nucl. Phys. B **563**, 63 (1999); A. E. Faraggi and M. Pospelov, Phys. Lett. B **458**, 237 (1999); A. K. Das and O. C. W. Kong, Phys. Lett. B **470**, 149 (1999); R. N. Mohapatra, S. Nandi and A. Perez-Lorenzana, Phys. Lett. B **466**, 115 (1999); A. Ioannisian and A. Pilaftsis, Phys. Rev. D **62**, 066001 (2000); R. N. Mohapatra and A. Perez-Lorenzana, Nucl. Phys. B **576**, 466 (2000); Nucl. Phys. B **593**, 451 (2001); Phys. Rev. D **67**, 075015 (2003); A. Ioannisian and J. W. F. Valle, Phys. Rev. D **63**, 073002 (2001); R. Barbieri, P. Creminelli and A. Strumia, Nucl. Phys. B **585**, 28 (2000); E. Ma, M. Raidal and U. Sarkar, Phys. Rev. Lett. **85**, 3769 (2000); E. Ma, G. Rajasekaran and U. Sarkar, Phys. Lett. B **495**, 363 (2000); G. C. McLaughlin and J. N. Ng, Phys. Lett. B **493**, 88 (2000); A. Lukas, P. Ramond, A. Romanino and G. G. Ross, Phys. Lett. B **495**, 136 (2000); JHEP **0104**, 010 (2001); D. O. Caldwell, R. N. Mohapatra and S. J. Yellin, Phys. Rev. Lett. **87**, 041601 (2001); Phys. Rev. D **64**, 073001 (2001); N. Cosme, J. M. Frere, Y. Gouverneur, F. S. Ling, D. Monderen and V. Van Elewyck, Phys. Rev. D **63**, 113018 (2001); K. Agashe and G. H. Wu, Phys. Lett. B **498**, 230 (2001); C. S. Lam and J. N. Ng, Phys. Rev. D **64**, 113006 (2001); A. De Gouvea, G. F. Giudice, A. Strumia and K. Tobe, Nucl. Phys. B **623**, 395 (2002); J. M. Frere, M. V. Libanov and S. V. Troitsky, JHEP **0111**, 025 (2001); C. S. Lam, Phys. Rev. D **65**, 053009 (2002); H. Davoudiasl, P. Langacker and M. Perelstein, Phys. Rev. D **65**, 105015 (2002); Q. H. Cao, S. Gopalakrishna and C. P. Yuan, Phys. Rev. D **69**, 115003 (2004); Phys. Rev. D **70**, 075020 (2004); J. L. Hewett, P. Roy and S. Roy, Phys. Rev. D **70**, 051903 (2004); E. Dudas, C. Grojean and S. K. Vempati, arXiv:hep-ph/0511001; M. T. Eisele and N. Haba, arXiv:hep-ph/0603158.
 - [9] K. R. Dienes and I. Sarcevic, Phys. Lett. B **500**, 133 (2001).
 - [10] N. Arkani-Hamed and M. Schmaltz, Phys. Rev. D **61**, 033005 (2000); N. Arkani-Hamed, Y. Grossman and M. Schmaltz, Phys. Rev. D **61**, 115004 (2000); E. A. Mirabelli and M. Schmaltz, Phys. Rev. D **61**, 113011 (2000).

- [11] See, *e.g.*, G. R. Dvali and M. A. Shifman, Phys. Lett. B **475**, 295 (2000); Y. Grossman, Int. J. Mod. Phys. A **15**, 2419 (2000); H. Georgi, A. K. Grant and G. Hailu, Phys. Rev. D **63**, 064027 (2001); T. G. Rizzo, Phys. Rev. D **64**, 015003 (2001); S. Nussinov and R. Shrock, Phys. Lett. B **526**, 137 (2002); H. C. Cheng, C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D **64**, 065007 (2001); N. Maru, Phys. Lett. B **522**, 117 (2001); D. E. Kaplan and T. M. P. Tait, JHEP **0111**, 051 (2001); W. F. Chang, I. L. Ho and J. N. Ng, Phys. Rev. D **66**, 076004 (2002); N. Haba and N. Maru, Phys. Rev. D **66**, 055005 (2002); Y. Grossman and G. Perez, Phys. Rev. D **67**, 015011 (2003); W. F. Chang and J. N. Ng, JHEP **0212**, 077 (2002); C. Biggio, F. Feruglio, I. Masina and M. Perez-Victoria, Nucl. Phys. B **677**, 451 (2004); B. Lillie and J. L. Hewett, Phys. Rev. D **68**, 116002 (2003); K. w. Choi, I. W. Kim and W. Y. Song, Nucl. Phys. B **687**, 101 (2004); B. Lillie, JHEP **0312**, 030 (2003); Y. Nagatani and G. Perez, JHEP **0502**, 068 (2005).
- [12] Y. Grossman, R. Harnik, G. Perez, M. D. Schwartz and Z. Surujon, Phys. Rev. D **71**, 056007 (2005); R. Harnik, G. Perez, M. D. Schwartz and Y. Shirman, JHEP **0503**, 068 (2005).
- [13] G. C. Branco, A. de Gouvea and M. N. Rebelo, Phys. Lett. B **506**, 115 (2001).
- [14] G. Barenboim, G. C. Branco, A. de Gouvea and M. N. Rebelo, Phys. Rev. D **64**, 073005 (2001).
- [15] J. M. Frere, G. Moreau and E. Nezri, Phys. Rev. D **69**, 033003 (2004); H. V. Klapdor-Kleingrothaus and U. Sarkar, Phys. Lett. B **541**, 332 (2002).
- [16] G. Moreau and J. I. Silva-Marcos, arXiv:hep-ph/0507145.
- [17] P. Q. Hung, Phys. Rev. D **67**, 095011 (2003).